Skoda's Theorem

Outline: 1) Statements
2) Integral Closure of Ideals
3) Proof of Theorems.

Notation: X (mostly) smooth variety / $\mathbb{C}$
$\dim X = n$

$\mathfrak{a} \subset \mathcal{O}_X$ ideal sheaf

L integral divisor on X

Nadel Vanishing (ideals version):

X projective. L, A divisors such that

→ $L - cA$ is big and nef $(c > 0)$

→ $\mathfrak{a} \otimes \mathcal{O}_X(A)$ is globally generated

Then, $H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(\mathfrak{a}^c)) = 0$

Let $k \in \mathbb{N}$, $k \geq c$

Key Point: If we pick $D_1, \ldots, D_k$ to be general sections of $\mathfrak{a} \otimes \mathcal{O}_X(A)$

then
$$D \sim_{\mathbb{Q}} \frac{1}{k}(D_1 + \cdots + D_k)$$
$$\mathcal{J}(\mathfrak{a}^c) = \mathcal{J}(cD)$$

I. Skoda's Theorem (local version)

X, $\mathfrak{a}$ as before. If $c \geq n = \dim X$
$$\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{c-1})$$

In particular, if $m \geq n$
$$\mathcal{J}(\mathfrak{a}^m) \subset \mathfrak{a}^{m-n+1}$$

Global Division Theorem:

X proj. and P big and nef divisor on X. L - integral divisor

$m \geq n+1$

$\mathcal{O}_X(L) \otimes \mathfrak{a}$ is globally generated by $s_1, \dots, s_p$.

then any section

$$s \in \Gamma(X, \mathcal{O}_X(K_X + mL + P) \otimes \mathcal{J}(\mathfrak{a}^m))$$

can be written as

$$s = \sum_{i=1}^{p} s_i h_i \quad \text{for}$$

$$h_i \in \Gamma(X, \mathcal{O}_X(K_X + (m-1)L + P) \otimes \mathcal{J}(\mathfrak{a}^{m-1}))$$

Remarks:

1) This is a generalization of Castelnuovo-Mumford regularity:

L - ample and globally generated

$$\mathfrak{a} = \mathcal{O}_X$$

$$\Rightarrow H^0(X, \mathcal{O}_X(L)) \otimes H^0(X, \mathcal{O}_X(K_X + (m-1)L + P))$$
$$\longrightarrow H^0(X, \mathcal{O}_X(K_X + mL + P))$$

is surjective

2) The bounds in both the statements are sharp: $\mathfrak{a} = \mathfrak{m}_x$ x-closed point

$$\begin{aligned} \mathcal{J}(\mathfrak{m}^k) &= \mathcal{O}_X & &\text{for } k < n \\ &= \mathfrak{m} & & k = n \\ &= \mathfrak{m}^2 & & k = n+1 \\ &\vdots & & \end{aligned}$$

$X = \mathbb{P}^n$, $L = \mathcal{O}(1)$, $P = \mathcal{O}(1)$

II. Integral Closure of Ideals

X - normal integral variety $\qquad \mathfrak{a} \subset \mathcal{O}_X$ ideal

$\nu : X^+ \longrightarrow X$

X^+: normalized blowup of $\mathfrak{a}$

$\mathfrak{a} \cdot \mathcal{O}_{X^+} = \mathcal{O}_{X^+}(-E)$ $\qquad E$ effective Cartier divisor on X^+

Then, $\overline{\mathfrak{a}} := \nu_* \mathcal{O}_{X^+}(-E) \subset \mathcal{O}_X$

$\overline{\mathfrak{a}}$: integral closure of $\mathfrak{a}$

An ideal $\mathfrak{a}$ is called integrally closed if

$\overline{\mathfrak{a}} = \mathfrak{a}$.

Rees Valuations: $E = \sum_{i=1}^{t} a_i E_i$ $\quad E_i$ - prime

then the valuations $\mathrm{ord}_{E_i}(-)$

are called Rees valuations of a.

The set $\left\{ \begin{matrix} \text{centers of } E_i \\ \text{on } X \end{matrix} \right\} = \bigcup_{n \geq 1} \left\{ \begin{matrix} \text{Associated points} \\ \text{of } \overline{a^n} \end{matrix} \right\}$

Primary Decomposition: Integrally closed ideals have a globally defined, canonical primary decomposition:

$$E = \sum_{i=1}^{t} a_i E_i \quad \text{and let}$$

$$q_i = \nu_* \mathcal{O}_{X^+}(-a_i E_i)$$

$$a = \overline{a} = q_1 \cap \dots \cap q_t \quad \text{and each}$$

q_i is primary

Prop: $f : Y \longrightarrow X$ proper birational map
Y, X normal.

$D \subset Y$ effective Cartier divisor

Then, $\mathfrak{a} = f_* \mathcal{O}_Y(-D)$ is integrally closed.

Proof: Take $\mu : Y' \longrightarrow Y$ such that

$$\mathfrak{a}\mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-D-E)$$

E - effective Cartier on Y'

$$\mu^* D \leq D + E$$

$$\overline{\mathfrak{a}} = (\mu \circ f)_* \mathcal{O}_{Y'}(-D-E) \subseteq \mathfrak{a}$$

$$\Rightarrow \mathfrak{a} = \overline{\mathfrak{a}}$$

$\square$

- All multiplier ideals are integrally closed.

Note: $\mathcal{J}(\mathfrak{a}^c) = \mathcal{J}(\overline{\mathfrak{a}}^c)$

In particular $\mathfrak{a} \subset \overline{\mathfrak{a}} \subset \mathcal{J}(\mathfrak{a})$

Characterizations of integral closure:

X affine, $\mathfrak{a} \subset \mathbb{C}[X]$ ideal

For $f \in \mathbb{C}[X]$, TFAE:

i) $f \in \overline{\mathfrak{a}}$

ii) f satisfies an equation of the form

$$f^k + b_1 f^{k-1} + \dots + b_k = 0$$

$$b_i \in \mathfrak{a}^i$$

iii) There exists a non-zero ideal $\mathfrak{b}$ s.t.

$$f \cdot \mathfrak{b} \subset \mathfrak{a} \cdot \mathfrak{b}$$

iv) There exists a non-zero element

$c \in \mathbb{C}[X]$ s.t.

$$c \cdot f^{l} \in \mathfrak{a}^{l} \quad \text{for all } l \gg 0$$

Brianson-Skoda Theorem: R regular local ring, $\mathfrak{a} \subset R$ ideal. Then $(n = \dim R)$

$m \geq n$ $$\overline{\mathfrak{a}^{m}} \subset \mathfrak{a}^{m-n+1}$$

Minimal Reductions: An ideal $\mathfrak{r} \subset \mathfrak{a}$ is called a reduction of $\mathfrak{a}$ if

$\overline{\mathfrak{r}} = \overline{\mathfrak{a}}$. It is called a minimal reduction

if $\mathfrak{r}$ can be generated by n elements.

$(n = \dim R)$

Prop: Minimal reductions exist!

Pf (Sketch): Pick n general linear combinations of generators of $\mathfrak{a}$

Similarly, if X proj, L integral Cartier, $s_1, \dots, s_p \in \Gamma(X, \mathcal{O}_X(L) \otimes \mathfrak{a})$ generating $\mathfrak{a} \otimes \mathcal{O}_X(L)$. Then, $n+1$ general linear combinations of s_i's generate $\mathcal{O}_X(L) \otimes \mathfrak{r}$ $\quad \mathfrak{r} \subset \mathfrak{a}$ reduction

Pf: $\nu : X^+ \longrightarrow X$ normalized blow-up of $\mathfrak{a}$

$$\nu^{-1}(\mathfrak{a}) = \mathcal{O}_{X^+}(-E)$$

Then $\nu^* s_i$'s generate $\mathcal{O}_{X^+}(\nu^* L - E)$

So $n+1$ general linear combinations,

$t_1, \dots, t_{n+1}$, will still generate

$$\mathcal{O}_{X^+}(\nu^*L - E) \quad \text{(after pulling back)}$$

$\Longrightarrow$ $t_1, \dots, t_{n+1}$ will generate

$$\mathcal{O}_X(L) \otimes \mathfrak{r} \qquad \underset{\text{reduction}}{\mathfrak{r} \subset \mathfrak{a}}$$

$\square$

III. Proofs.

Skoda complexes:

X smooth $\qquad \mathfrak{a} \subset \mathcal{O}_X \qquad L$

$$s_1, \dots, s_p \in \Gamma(X, \mathcal{O}_X(L) \otimes \mathfrak{a})$$

generating sections

$V = \mathbb{C}\langle s_1, \dots, s_p \rangle \qquad c \geq 0 \qquad m \geq n$

Note: we have $\mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{m-1}) \subset \mathcal{J}(\mathfrak{a}^m) \quad \forall m \geq 1$

$$0 \to \Lambda^p V \otimes \mathcal{J}(\mathfrak{a}^{c+m-p}) \otimes \mathcal{O}_X(-pL) \to \cdots \to \Lambda^2 V \otimes \mathcal{J}(\mathfrak{a}^{c+m-2}) \otimes \mathcal{O}_X(-2L) \to V \otimes \mathcal{J}(\mathfrak{a}^{m-1+c}) \otimes \mathcal{O}_X(-L) \to \mathcal{J}(\mathfrak{a}^{c+m}) \to 0$$

(m^{th} - Skoda complex - Skod_m)

Theorem: Skod_m is exact if $m \geq p$

Pf: Consider a log-resolution of $\mathfrak{a}$

$$\mu : X' \longrightarrow X \qquad \mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-A)$$

since $s_1, \ldots, s_p$ generate $\mathcal{O}_X(L) \otimes \mathfrak{a}$

$$s_i' = \mu^* s_i \quad \text{generate } \mathcal{O}_{X'}(\mu^* L - A)$$

$$\Lambda_m = K_{X'/X} - mA - \lfloor cA \rfloor$$

Consider Koszul complex determined by s_i'.

$$0 \to \Lambda^p V \otimes \mathcal{O}_{X'}(p(A-\mu^*L)) \to \cdots \to \Lambda^2 V \otimes \mathcal{O}_{X'}(2(A-\mu^*L)) \to V \otimes \mathcal{O}_{X'}(A-\mu^*L) \to \mathcal{O}_{X'} \to 0$$

This is exact b/c $V = \langle s_1', \dots, s_p' \rangle$ generate $\mathcal{O}_{X'}(\mu^*L - A)$

Twist this by Λ_m, to get

$$0 \to \Lambda^p V \otimes \mathcal{O}_{X'}(\Lambda_{m-p} - p\mu^*L) \to \cdots \to V \otimes \mathcal{O}_{X'}(\Lambda_{m-1} - \mu^*L) \to \mathcal{O}_{X'}(\Lambda_m) \to 0$$

Note: For $j>0$, $R^j \mu_* \mathcal{O}_{X'}(A_{m-i} - i\mu^* L) = 0$ by local vanishing

Skod_m is just got by applying μ_* to this. Moreover, all terms in the complex have no $R^i \mu_*(\ )$ for $i>0$

(local vanishing theorem)

This proves the exactness of Skod_m.

Proofs of Main Theorems:

Skoda's Theorem:

Recall: $c \geq n = \dim X$

$$\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{c-1})$$

Pf: The question is local, so assume X affine

Enough to prove that

$$\mathcal{J}(\mathfrak{a}^c) \subseteq \mathfrak{r} \cdot \mathcal{J}(\mathfrak{a}^{c-1}) \subseteq \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{c-1})$$

$$(\mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{c-1}) \subset \mathcal{J}(\mathfrak{a}^c) \text{ always true})$$

So, we may replace $\mathfrak{a}$ by $\mathfrak{r}$ and assume

$\mathfrak{a}$ is generated by $\leq n$ elements.

Consider the ~~l~~ast map:

$$V \otimes \mathcal{J}(\mathfrak{a}^{c+m-1}) \longrightarrow \mathcal{J}(\mathfrak{a}^{c+m}) \to 0$$

is surjective.

$L = \mathcal{O}_X$ $\square$

$m = \lfloor c \rfloor \quad c = \{c\}$

Proof (Global Division Theorem): X proj

P - nef and big divisor, L - integral

$m \geq n+1$ and $s_1, \dots, s_p$ generate

$\mathcal{O}_X(L) \otimes \mathfrak{a}$.

$V = \mathbb{C}\langle s_1, \dots, s_p \rangle$

W.T.S.

$$V \otimes \Gamma(X, \mathcal{O}_X(K_X + (m-1)L + P) \otimes \mathcal{J}(\mathfrak{a}^{m-1}))$$

$$\longrightarrow \Gamma(X, \mathcal{O}_X(K_X + mL + P) \otimes \mathcal{J}(\mathfrak{a}^m))$$

is surjective

Note: enough to check this for

$t_1, \dots, t_{n+1}$ (general linear combinations of s_i's)

Again consider the Skod_m and twist it by $\mathcal{O}_X(K_X + mL + P)$

$$\cdots \to \Lambda^2 V \otimes \mathcal{O}_X(K_X + (m-2)L + P) \otimes \mathcal{J}(\mathfrak{a}^{m-2}) \to V \otimes \mathcal{O}_X(K_X + (m-1)L + P) \otimes \mathcal{J}(\mathfrak{a}^{m-1}) \to \mathcal{O}_X(K_X + mL + P) \otimes \mathcal{J}(\mathfrak{a}^m) \to 0$$

Now, taking $H^0(X, \text{—})$ of the above complex remains exact, due to Nadel Vanishing: $H^i(X, \mathcal{O}_X(K_X + (m-i)L + P) \otimes \mathcal{J}(\mathfrak{a}^{m-i})) = 0$

(because $\mathfrak{a} \otimes \mathcal{O}_X(L)$ is globally generated)

This completes the proof! $\boxtimes$